What's a more useful way to think about the nullspace? "The vectors a matrix sends to zero" is technically correct but is more of a mechanical byproduct we use to describe a deeper property.

1 Checking injectivity of a linear map

Let $T : V \to W$ be a linear map for vector spaces V, W. Naturally, we want to know if it's injective.

- The nullspace is a tool to describe the injectivity of T.
- Essentially, it describes all such a, b such that the injectivity condition $T(a) = T(b) \implies a = b$ is false.

What does that have anything to do with "the set of all x such that $T(x) = 0_W$" though? Seems unrelated...

- But this is actually just a simpler injectivity test that utilizes linearity.
- Observe (LHS):

$$\begin{aligned} T(a) &= T(b) \\ T(a) - T(b) &= 0_W \\ T(a - b) &= 0_W \end{aligned}$$

- Also (RHS):

$$\begin{aligned} a &= b \\ a - b &= 0_V \end{aligned}$$

- New injectivity condition: $T(a - b) = 0_W \implies a - b = 0_V$.
- Since $f(a, b) = a - b$ is surjective on V (vector space property), reparameterize WLOG: $x = a - b$.
- New clean injectivity condition: $T(x) = 0_W \implies x = 0_V$.

So the **nullspace** $\mathcal{N}(T)$ is all x such that $T(x) = 0_W$, and if $\mathcal{N}(T) \neq \{0_V\}$, then the injectivity condition $T(x) = 0_W \implies x = 0_V$ is **violated**.

2 Checking shape of injectivity of a linear map

And tracking the set of inputs $\mathcal{N}(T)$ that violate injectivity has another benefit: this is how injectivity is violated for every output in W, not only 0_W. We get this because of linearity (it's really a great property).

- Proof: say we want to show $\forall (x, n) \in V \times V, T(x) = T(x + n) \iff n \in \mathcal{N}(T)$.
- This is borderline trivial by linearity:

$$\begin{aligned} T(x + n) &= T(x) \\ T(x + n) - T(x) &= 0_W \\ T((x + n) - x) &= 0_W \\ T(n) &= 0_W \end{aligned}$$

- So $T(n) = 0_W \iff n \in \mathcal{N}(T)$. This is literally the definition of the nullspace we're operating under. QED

So not only does $\mathcal{N}(T)$ tell us if T is injective ($\mathcal{N}(T) = \{0\}$), but it also acts as a stencil to find *all* solutions $x \in V : T(x) = b$, as long as we're given a single particular solution $p \in V : T(p) = b$ we can "place our stencil at". To make it more concrete:

- If the nullspace is a line, we know the failure of injectivity for any output is a line's worth of inputs.
- If the nullspace is a plane, we know the failure of injectivity for any output is a plane's worth of inputs.
- If the nullspace is some N-dimensional vector space, we know the failure of injectivity for any output is that same N-dimensional vector space shape somewhere in $\text{dom}(T)$.