1 Problem

We have a system of differential equations $x_1, x_2, x_3 \in C^\infty(\mathbb{R})$:

$$x_1' = 3x_1 + x_2$$
$$x_2' = x_2$$
$$x_3' = 4x_1 + 2x_2 + x_3$$

How do we find x_1, x_2, x_3 for initial conditions $x_1(0) = -3, x_2(0) = 1, x_3(0) = -1$?

2 Solution

2.1 Idea

Recall that if we have $f' = kf$ ($f \in C^\infty(\mathbb{R})$), the solution set is trivial: $f = Ce^{kt}$ for all $C \in \mathbb{R}$. We want to break the problem down into blocks that look like $f' = kf$ somehow; it may already feel intuitive that eigenvectors would probably be useful here, even if not immediately clear exactly how.

Now let's write our system as $x' = Ax$ where $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. So $x, x' \in C^\infty(\mathbb{R}, \mathbb{R}^3)$, which is still a vector space. The idea is that the set of solutions x to this system $S \subset C^\infty(\mathbb{R}, \mathbb{R}^3)$ is also a subspace of $C^\infty(\mathbb{R}, \mathbb{R}^3)$:

- **Zero vector:** for $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $0 = A0$ is true. Therefore $0 \in S$.
- **Addition is closed:** for $x, y \in S$, $Ax = x'$ and $Ay = y'$ so $Ax + Ay = A(x + y) = x' + y'$. Therefore $x + y \in S$.
- **Scaling is closed:** for $x \in S, c \in \mathbb{R}$, $Ax = x' \implies cAx = cx' \implies A(cx) = (cx)'$. Therefore $cx \in S$.

So if we can find a basis of functions $f_1, ..., f_n \in S$ for S, we have a closed form for any element in S: $x = \sum_{i=1}^{n} c_i f_i$ for $c_i \in \mathbb{R}$. It's then easy to find the particular solution the initial conditions ask for; just plug the values in.

- In general, the solution space $S \subset C^\infty(\mathbb{R}, \mathbb{R}^n)$ has dimension n. Why?
- *Motivation*: Notice that x is uniquely identifiable by $x(0)$ (or any "initial state"; we arbitrarily use 0). Because $x'(t) = Ax(t)$ for all t, the movement of x is entirely defined across its entire domain, and consequently x is entirely defined across its entire domain, simply by its starting position in $\mathbb{R}^n$. This might suggest that there could be a linear isomorphism between S and $\mathbb{R}^n$, though this is only the injectivity condition; we'll have to dig deeper. We'll construct a linear bijection $E : S \to \mathbb{R}^n$ where $E(x) = x(0)$ to show that $\dim(S) = \dim(\mathbb{R}^n)$.
  - **Linear**: $E(x + y) = (x + y)(0) = x(0) + y(0) = E(x) + E(y)$ and $E(cx) = (cx)(0) = c(x(0)) = cE(x)$.
  - **Injective**: We just showed this previously; we know it is.
  - **Surjective**: every $v \in \mathbb{R}^n$ needs to produce a $x \in S$. For any v, we can use $x = e^{At}v$ to wrangle out a corresponding $x \in S$; membership in S requires $x' = Ax$, which this satisfies ($x' = Ae^{At}v = Ax$).
- Therefore, $S \cong \mathbb{R}^n \implies \boxed{\dim(S) = n}$.

2.2 Finding the basis

Obviously, this basis decomposition idea only makes sense if the basis functions f_i are easier to find than the original x function.

- Luckily, eigenvectors make f_i easy to define; if v_i is the ith eigenvector of A, we make f_i be a function that only outputs elements in $\text{span}(v_i)$; that is, $f_i(t) = g_i(t)v_i$ for some scalar function $g_i \in C^\infty(\mathbb{R})$. If A has an eigenbasis, then all f_i will be LI and thus form a basis for S. If A doesn't have an eigenbasis... well, I haven't learned enough to know what to do then, but this method is obviously inapplicable.
- Now take a particular solution $x_p = f_i(t)$ (expanded, we have $x_p = g_i(t)v_i$); for it to satisfy $x_p' = Ax_p$, $g_i(t)$ must satisfy certain constraints. Specifically:

$$x_p' = Ax_p$$
$$g_i'(t)v_i = A(g_i(t)v_i)$$
$$g_i'(t)v_i = g_i(t)\lambda v_i$$
$$g_i'(t) = \lambda g_i(t)$$

- So for $x_p' = Ax_p$, it is necessary that g_i satisfies $g_i'(t) = \lambda_i g_i(t)$. Perfect — this is trivial now. Set $g_i(t) = e^{\lambda_i t}$. (We choose to leave the constant C as 1 because the constant is arbitrary in the presence of an eigenvector in this context; furthermore, the scale doesn't matter anyways, $f_i(t)$ is going to be a basis function regardless of how much it gets scaled.)
- Thus, $\boxed{f_i(t) = e^{\lambda_i t} v_i}$.

2.3 Actually solving the problem now

If you do the math and find the (λ_i, v_i) pairs for our problem, the basis is:

- $f_1(t) = e^{3t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$
- $f_2(t) = e^{t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$
- $f_3(t) = e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So all solutions x which satisfy $x' = Ax$ live in $\text{span}(f_1, f_2, f_3)$; in other words, any particular solution x is of the form $x = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_3 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for coefficients $c_1, c_2, c_3 \in \mathbb{R}$.

Plug in the three initial conditions into the equation to nail down a single point in the span:

- $\begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- This gives us a system of equations where $c_1 = -\frac{5}{2}, c_2 = -\frac{1}{2}, c_3 = 4$.
- So $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -\frac{5}{2} e^{3t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{1}{2} e^{t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 4e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

The final answer is, extracting from the vectorized solution:

- $\boxed{x_1(t) = -\frac{5}{2} e^{3t} - \frac{1}{2} e^{t}}$
- $\boxed{x_2(t) = e^{t}}$
- $\boxed{x_3(t) = -5e^{3t} + 4e^{t}}$