Notes on the approach I figured out (before learning about $P_{\bar{B}}^{-1}AP_B$) to finding the coordinate matrix so I don't forget.

1 Problem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(x) = \begin{pmatrix} x+2y \\ 3x-y \end{pmatrix}$.

The domain basis is $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ and the codomain basis is $\bar{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Find the coordinate matrix M such that $\forall x \in \mathbb{R}^2, [T(x)]_{\bar{B}} = M \cdot [x]_B$.

2 Solution

The idea is to find enough independent constraints on M to uniquely construct it by analyzing T's behavior. Specifically, we choose inputs to T such that we specify one column of M at a time. We then need one constraint per column of M.

Since we want to isolate columns, we choose the "standard basis" in the B coefficient vectors. For example in $\mathbb{R}^2$, we want to isolate $M \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $M \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to get M_1 and M_2.

- So, we should choose inputs $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ because $\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Now to analyze T with these inputs we've chosen...

- **Column 1**: $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. By inspection, $\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]_{\bar{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- **Column 2**: $T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$. By inspection, $\left[\begin{pmatrix} -1 \\ 4 \end{pmatrix}\right]_{\bar{B}} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$.

So we have two constraints on M now:

- $\left[T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right]_{\bar{B}} = M \cdot \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_B$
- $\left[T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right]_{\bar{B}} = M \cdot \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]_B$

Simplify:

- $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = M \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $\begin{pmatrix} -5 \\ 4 \end{pmatrix} = M \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So $\boxed{M = \begin{pmatrix} 1 & -5 \\ 2 & 4 \end{pmatrix}}$.

3 The "official" formula

$M = P_{\bar{B}}^{-1}AP_B$ is simple, actually — it essentially takes in $[x]_B$ from the domain, transforms it to x using P_B, maps it to the codomain using A, then turns it into a coordinate vector again with $P_{\bar{B}}^{-1}$.

I guess the previous approach is more generalizable than the matrix formula as this kind of assumes T is of the form Ax. Not sure if there's a direct way to obtain A for something like a polynomial map. Either way, it's another middleman that I like to not have to go through.